Exercise 1I

1 a $f(z) = z^3 - 6z^2 + 21z - 26$

$f(2) = (2)^3 - 6(2)^2 + 21(2) - 26$
$= 8 - 24 + 42 - 26$
$= 0$

b If $f(2) = 0$ then $z - 2$ is a factor of f(z)

$$\begin{array}{r|l} & z^2 - 4z + 13 \\ \hline z-2 & z^3 - 6z^2 + 21z - 26 \\ & z^3 - 2z^2 \\ & -4z^2 + 21z \\ & -4z^2 + 8z \\ & 13z - 26 \\ & 13z - 26 \\ & 0 \end{array}$$

So $0 = f(z)$
$= z^3 - 6z^2 + 21z - 26$
$= (z-2)(z^2 - 4z + 13)$

Either $(z-2) = 0$ or $(z^2 - 4z + 13) = 0$

Solve $z^2 - 4z + 13 = 0$ by completing the square:

$(z-2)^2 - 4 + 13 = 0$
$(z-2)^2 = -9$
$z - 2 = \pm 3\mathrm{i}$
$z = 2 \pm 3\mathrm{i}$

So the roots of $z^3 - 6z^2 + 21z - 26 = 0$ are $2,\ 2 + 3\mathrm{i}$ and $2 - 3\mathrm{i}$

2 a $f(z) = 2z^3 + 5z^2 + 9z - 6$

$f\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^3 + 5\left(\frac{1}{2}\right)^2 + 9\left(\frac{1}{2}\right) - 6$
$= \frac{2}{8} + \frac{5}{4} + \frac{9}{2} - 6$
$= \frac{1}{4} + \frac{5}{4} + \frac{18}{4} - \frac{24}{4}$
$= 0$

2 b $f\left(\frac{1}{2}\right) = 0 \Rightarrow\ 2z - 1$ is a factor of $f(z) = 0$

Hence

$f(z) = 2z^3 + 5z^2 + 9z - 6$
$= (2z-1)(z^2 + bz + c)$
$= 2z^3 + (2b-1)z^2 + (2c-b)z - c$

Equating z^2 terms:
$2b - 1 = 5$, so $b = 3$

Equating constant terms:
$-c = -6$, so $c = 6$

Hence $f(z) = (2z-1)(z^2 + 3z + 6)$

c $f(z) = 0 \Rightarrow$ Either $(2z-1) = 0$
or $(z^2 + 3z + 6) = 0$

Solve $(z^2 + 3z + 6) = 0$ by completing the square:

$\left(z + \frac{3}{2}\right)^2 - \frac{9}{4} + 6 = 0$

$\left(z + \frac{3}{2}\right)^2 = -\frac{15}{4}$

$z + \frac{3}{2} = \pm \frac{\sqrt{15}}{2}\mathrm{i}$

$z = -\frac{3}{2} \pm \frac{\sqrt{15}}{2}\mathrm{i}$

So the roots of $2z^3 + 5z^2 + 9z - 6 = 0$ are $\frac{1}{2}$, $\left(-\frac{3}{2} + \frac{\sqrt{15}}{2}\mathrm{i}\right)$, and $\left(-\frac{3}{2} - \frac{\sqrt{15}}{2}\mathrm{i}\right)$

3 $g(z)=2z^3-4z^2-5z-3$

$z=3$ is a root of $g(z)=0$

$\Rightarrow z-3$ is a factor of $g(z)$

$$\begin{array}{r} 2z^2+2z+1 \\ z-3\overline{)2z^3-4z^2-5z-3} \\ 2z^3-6z^2 \\ 2z^2-5z \\ 2z^2-6z \\ z-3 \\ z-3 \\ 0 \end{array}$$

So $g(z)=(z-3)(2z^2+2z+1)=0$

Solve $2z^2+2z+1=0$ using the quadratic formula:

$$z=\frac{-2\pm\sqrt{(2)^2-(4)(2)(1)}}{2(2)}$$

$$z=\frac{-2\pm\sqrt{4-8}}{4}$$

$$z=\frac{-2\pm 2\mathrm{i}}{4}$$

$$z=-\tfrac{1}{2}\pm\tfrac{1}{2}\mathrm{i}$$

So the roots of $2z^3-4z^2-5z-3=0$ are 3, $-\frac{1}{2}+\frac{1}{2}\mathrm{i}$ and $-\frac{1}{2}-\frac{1}{2}\mathrm{i}$

4 a $p(z)=z^3+4z^2-15z-68$

$z=-4+\mathrm{i}$ is a solution to $p(z)=0$, so

$z=-4-\mathrm{i}$ is also a solution to $p(z)=0$.

Hence $(z-(-4+\mathrm{i}))(z-(-4-\mathrm{i}))$ is a factor of $p(z)$

$$(z-(-4+\mathrm{i}))(z-(-4-\mathrm{i}))$$
$$=z^2-(-4-\mathrm{i})z-(-4+\mathrm{i})z+(-4+\mathrm{i})(-4-\mathrm{i})$$
$$=z^2+4z+\mathrm{i}z+4z-\mathrm{i}z+16+4\mathrm{i}-4\mathrm{i}-\mathrm{i}^2$$
$$=z^2+8z+17$$

$\therefore z^2+8z+17$ is a factor of $p(z)$

b $0=p(z)$

$$=z^3+4z^2-15z-68$$
$$=(z^2+8z+17)(az+b)$$
$$=az^3+(b+8a)z^2+(17a+8b)z+17b$$

Equate z^3 coefficients:

$a=1$

Equate constants:

$17b=-68\Rightarrow b=-4$

Hence $z-4$ is a factor of $p(z)$ and $z=4$ is a solution of $p(z)=0$

So the roots of $z^3+4z^2-15z-68=0$ are 4, $-4+\mathrm{i}$ and $-4-\mathrm{i}$

5 a $z^3+9z^2+33z+25=(z+1)(z^2+az+b)$

$$=z^3+(1+a)z^2+(a+b)z+b$$

Equate z^2 terms:

$a+1=9$, so $a=8$

Equate constant terms:

$b=25$

b $f(z)=(z+1)(z^2+8z+25)=0$

Either $z+1=0\Rightarrow z=-1$ **or** $z^2+8z+25=0$

Solve $z^2+8z+25=0$ by completing the square:

$$(z+4)^2-16+25=0$$
$$(z+4)^2=-9$$
$$z+4=\pm 3i$$
$$z=-4\pm 3i$$

So the roots of $z^3+9z^2+33z+25=0$ are -1, $-4+3\mathrm{i}$ and $-4-3\mathrm{i}$

c $-1+(-4+3\mathrm{i})+(-4-3\mathrm{i})=-9$

6 a If $3+\mathrm{i}$ is a root, then $3-\mathrm{i}$ is also a root.

6 b $\mathrm{g}(z)=(z-6)(z-(3+\mathrm{i}))(z-(3-\mathrm{i}))$

$=(z-6)(z^2-(3-\mathrm{i})z-(3+\mathrm{i})z+(3+\mathrm{i})(3-\mathrm{i}))$

$=(z-6)(z^2-3z+\mathrm{i}z-3z-\mathrm{i}z+9-3\mathrm{i}+3\mathrm{i}-\mathrm{i}^2)$

$=(z-6)(z^2-6z+10)$

$=z^3-6z^2+10z-6z^2+36z-60$

$=z^3-12z^2+46z-60$

So $c=46$ and $d=-60$.

7 $\mathrm{h}(z)=2z^3+3z^2+3z+1$

$=(2z+1)(az^2+bz+c)$

$=2az^3+(a+2b)z^2+(2c+b)z+c$

Equate z^3 coefficient:

$2a=2$, so $a=1$

Equate z^2 coefficients:

$a+2b=3$, so $b=1$

Equate constants:

$c=1$

$0=h(z)$

$=(2z+1)(z^2+z+1)$

Either $2z+1=0\Rightarrow z=-\frac{1}{2}$ or

$z^2+z+1=0$

Solve $z^2+z+1=0$ by completing the square:

$\left(z+\frac{1}{2}\right)^2-\frac{1}{4}+1=0$

$\left(z+\frac{1}{2}\right)^2=-\frac{3}{4}$

$z+\frac{1}{2}=\pm\frac{\sqrt{3}}{2}\mathrm{i}$

$z=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}\mathrm{i}$

So the roots of $2z^3+3z^2+3z+1=0$ are

$-\frac{1}{2}$, $\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i}\right)$ and $\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}\right)$

8 a $\mathrm{f}(z)=z^3-6z^2+28z+k$

$0=\mathrm{f}(2)$

$=(2)^3-6(2)^2+28(2)+k$

$=8-24+56+k$

$k=-40$

b $z=2$ is a root $\Rightarrow z-2$ is a factor

$$\begin{array}{r|l} & z^2-4z+20 \\ \hline z-2 & z^3-6z^2+28z-40 \\ & z^3-2z^2 \\ & \quad -4z^2+28z \\ & \quad -4z^2+\ \ 8z \\ & \qquad\quad 20z-40 \\ & \qquad\quad 20z-40 \\ & \qquad\qquad\quad 0 \end{array}$$

$0=\mathrm{f}(z)$

$=(z-2)(z^2-4z+20)$

Either $z=2$ or $z^2-4z+20=0$

Solve $z^2-4z+20=0$ by completing the square:

$z^2-4z+20=0$

$(z-2)^2-4+20=0$

$(z-2)^2=-16$

$z-2=\pm4\mathrm{i}$

$z=2\pm4\mathrm{i}$

So the other roots of $z^3-6z^2+28z+40=0$ are $2+4\mathrm{i}$ and $2-4\mathrm{i}$.

9 $z^4-16=0$

$(z^2-4)(z^2+4)=0$

$(z-2)(z+2)(z^2+4)=0$

Either $z-2=0\Rightarrow z=2$

or $z+2=0\Rightarrow z=-2$

or $z^2+4=0\Rightarrow z^2=-4$

$\Rightarrow z=\pm2\mathrm{i}$

So the roots of $z^4-16=0$ are $2, -2, 2\mathrm{i}$ and $-2\mathrm{i}$

10 a $z^4 - 12z^3 + 31z^2 + 108z - 360$

$= (z^2 - 9)(z^2 + bz + c)$

$= z^4 + bz^3 + (-9 + c)z^2 - 9bz - 9c$

Equate z^3 coefficients:

$b = -12$

Equate constants:

$-9c = -360$, so $c = 40$

So $f(z) = (z^2 - 9)(z^2 - 12z + 40)$

b $0 = f(z)$

$= (z^2 - 9)(z^2 - 12z + 40)$

Either $z^2 - 9 = 0 \Rightarrow z = \pm 3$

Or $z^2 - 12z + 40 = 0$

$(z-6)^2 - 36 + 40 = 0$

$(z-6)^2 = -4$

$z - 6 = \pm 2\text{i}$

$z = 6 \pm 2\text{i}$

So the solutions to $f(z) = 0$ are

$3,\ -3,\ 6 + 2\text{i}$ and $6 - 2\text{i}$

11 If $g(2 + 3\text{i}) = 0$ then also $g(2 - 3\text{i}) = 0$

So $(z - (2 + 3\text{i}))(z - (2 - 3\text{i}))$ is a factor of $g(z)$

Now $(z - (2 + 3\text{i}))(z - (2 - 3\text{i}))$

$= z^2 - (2 - 3\text{i})z - (2 + 3\text{i})z + (2 + 3\text{i})(2 - 3\text{i})$

$= z^2 - 2z + 3\text{i}z - 2z - 3\text{i}z + 4 - 6\text{i} + 6\text{i} - 9\text{i}^2$

$= z^2 - 4z + 13$

So $z^2 - 4z + 13$ is a factor of $g(z)$

Use long division to find another factor:

$$
\begin{array}{r}
z^2 + 6z + 10 \\
z^2 - 4z + 13 \overline{) z^4 + 2z^3 - z^2 + 38z + 130} \\
z^4 - 4z^3 + 13z^2 \qquad\qquad \\
6z^3 - 14z^2 + 38z \qquad \\
6z^3 - 24z^2 + 78z \qquad \\
10z^2 - 40z + 130 \\
10z^2 - 40z + 130 \\
0
\end{array}
$$

11 So $g(z) = (z^2 - 4z + 13)(z^2 + 6z + 10)$

$g(z) = 0 \Rightarrow$

either $(z^2 - 4z + 13) = 0 \Rightarrow z = 2 \pm 3\text{i}$

(these were the two solutions stated at the beginning of the question)

or $z^2 + 6z + 10 = 0$

$(z+3)^2 - 9 + 10 = 0$

$(z+3)^2 = -1$

$z + 3 = \pm\text{i}$

$z = -3 \pm \text{i}$

So the roots of $g(z) = 0$ are

$2 + 3\text{i},\ 2 - 3\text{i},\ -3 + \text{i}$ and $-3 - \text{i}$

12 a If $z = 2 - 3\text{i}$ is a root then $z = 2 + 3\text{i}$ is also a root

So $(z - (2 - 3\text{i}))(z - (2 + 3\text{i}))$ is a factor of $f(z)$

Now $(z - (2 - 3\text{i}))(z - (2 + 3\text{i}))$

$= z^2 - (2 + 3\text{i})z - (2 - 3\text{i})z + (2 - 3\text{i})(2 + 3\text{i})$

$= z^2 - 2z - 3\text{i}z - 2z + 3\text{i}z + 4 + 6\text{i} - 6\text{i} - 9\text{i}^2$

$= z^2 - 4z + 13$

So $z^2 - 4z + 13$ is a factor of $f(z)$

$\therefore\ f(z) = z^4 - 10z^3 + 71z^2 + Qz + 442$

$= (z^2 - 4z + 13)(z^2 + bz + c)$

$= z^4 + (4 + b)z^3 + (c - 4b + 13)z^2 + (13b - 4c)z + 13c$

Equate z^3 coefficients:

$-4 + b = -10$, so $b = -6$

Equate constants:

$13c = 442$, so $c = 34$

Hence $(z^2 - 6z + 34)$ is a factor of $f(z)$

12 b To find Q, equate z coefficients:

$13b - 4c = Q$

$13(-6) - 4(34) = Q$

$Q = -214$

12 c $0 = \text{f}(z)$

$= (z^2 - 4z + 13)(z^2 - 6z + 34)$

either $(z^2 - 4z + 13) = 0 \Rightarrow z = 2 \pm 3\text{i}$

(these were the two solutions stated at the beginning of the question)

or $z^2 - 6z + 34 = 0$

$(z-3)^2 - 9 + 34 = 0$

$(z-3)^2 = -25$

$z - 3 = \pm 5\text{i}$

$z = 3 \pm 5\text{i}$

So the roots of $\text{f}(z) = 0$

are $2 - 3\text{i},\ 2 + 3\text{i},\ 3 + 5\text{i}$ and $3 - 5\text{i}$.

Challenge

$z^5 + bz^4 + cz^3 + dz^2 + ez + f = 0$ (*)

The other two roots of (*) are –2i and 1 – i

For any pair of complex conjugate roots α and β of (*), the quadratic expression

$(z-\alpha)(z-\beta) = z^2 - (\alpha+\beta)z + \alpha\beta$

is a factor of (*)

Using $\alpha = 2\text{i}$ and $\beta = -2\text{i}$,

$\alpha + \beta = 2\text{i} - 2\text{i} = 0$

$\alpha\beta = (2\text{i})(-2\text{i}) = -4\text{i}^2 = 4$

So one quadratic factor of (*) is

$z^2 + 4 = 0$

Using $\alpha = 1 + \text{i}$ and $\beta = 1 - \text{i}$,

$\alpha + \beta = (1+\text{i}) + (1-\text{i}) = 2$

$\alpha\beta = (1+\text{i})(1-\text{i})$

$= 1(1-\text{i}) + \text{i}(1-\text{i})$

$= 1 - \text{i} + \text{i} - \text{i}^2 = 2$

So a second quadratic factor of (*) is

$z^2 - 2z + 2 = 0$

Also, since –2 is the only real root of (*), then $(z+2)$ is a linear factor of (*)

Hence,

$0 = z^5 + bz^4 + cz^3 + dz^2 + ez + f$

$= (z+2)(z^2+4)(z^2-2z+2)$

$= (z^3 + 2z^2 + 4z + 8)(z^2 - 2z + 2)$

$= z^3(z^2 - 2z + 2) + 2z^2(z^2 - 2z + 2)$

$+ 4z(z^2 - 2z + 2) + 8(z^2 - 2z + 2)$

$= z^5 - 2z^4 + 2z^3 + 2z^4 - 4z^3 + 4z^2$

$+ 4z^3 - 8z^2 + 8z + 8z^2 - 16z + 16$

$= z^5 + 2z^3 + 4z^2 - 8z + 16$

$b = 0,\ c = 2,\ d = 4,\ e = -8,\ f = 16$